

# Complete Shaking Force and Shaking Moment Balancing of Mechanisms Using a Moving Rigid Body

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*This paper addresses the mass balancing of mechanisms using a single rigid body („balancing body“). Firstly, the expressions of dynamic forces and moments acting on the machine frame, which are caused by arbitrary planar and spatial mechanisms are established. The general balancing conditions are then derived. By motion control of the balancing body, any resultant inertia forces and moments of several mechanisms can also be fully compensated. The desired motion of the balancing body is calculated in order that the sum of inertia forces and moments of the mechanisms and the balancing body is zero. Theoretically, the balancing body is possible to compensate the dynamic loads of the machine frame, even if several mechanisms with any structure are located in the machine frame. The balancing theory of planar mechanisms is presented in more detail. Finally, the proposed balancing method is illustrated by a numerical example, in which three components of dynamic loads caused by a planar mechanism in the steady state are given as the time-periodic functions. It can be shown that the proposed approach is an alternative to the conventional balancing methods and especially applicable in practice with piezoelectric actuators.*

## 1 Introduction

Mass balancing of mechanisms brings about a reduction of the variable dynamic loads on the machine frame. In effect, this minimizes the noise and wear, and improves the dynamic performance of the mechanisms, (Arakelian and Smith, 2005). The objective of mass balancing is to completely eliminate or partially reduce the resultant inertia forces and moments caused by all moving links. A number of mass balancing solutions (mass redistribution, using counterweights or adding balancer-mechanisms as cams, gears, parallelogram chains, planetary gears, and etc.) are often used, see important references e.g. (Berkof, 1973), (Lowen et. al., 1983), (Kochev, 1990), (Dresig et. al., 1994), (Ye and Smith, 1994), (Esat and Bahai, 1999), (Wu and Gosselin, 2007), (Moore et. al., 2009). However, these solutions have the main disadvantage that increase the manufacturing cost and the complexity of the mechanisms, (Arakelian and Smith, 1999), (Kochev, 2000).

During the motion of a machine, dynamic loads caused by moving mechanisms are transmitted to the machine frame. If the machine frame is rigid and fixed to the ground, no dynamic loads is caused by itself. In this case, only the resultant inertia force and the resultant inertia moment from the mechanisms located on the machine frame are of interest. No dynamic loads are transmitted to the base if the resultant inertia force and the resultant inertia moment are completely eliminated, (Chaudhary and Saha, 2009).

By controlled movements of a single rigid body (called *balancing body*) we can create balancing forces and moments that compensate the dynamic loads from the mechanisms located on the same machine frame. The mechanisms can have arbitrary structures and perform arbitrary motions, which, however are constrained motions and described by functions of a single input coordinate or time. In addition, the mechanisms can be arranged with arbitrary spatial configurations in the machine frame. It is possible to calculate the desired motion of the balancing body to implement the balancing task. The calculation requires either the data of the geometrical and inertia parameters of all moving links, or measurements of the position-dependent resultant inertia forces and moments.

## 2 Balancing Theory

### 2.1 Dynamic loads acting on the machine frame-shaking force and shaking moment

We consider an arbitrary link of a single degree-of-freedom spatial mechanism as depicted in Fig. 1. The resultant inertia force (shaking force) and the resultant inertia moment (shaking moment) caused by all moving links of the mechanism can be expressed in the matrix form, (Nguyen and Nguyen, 2007)

$$\mathbf{F}^* = -\frac{d}{dt}\mathbf{p}, \quad \mathbf{M}_O^* = -\frac{d}{dt}\mathbf{l}_O \quad (1)$$

where  $\mathbf{p}$  is the linear momentum and  $\mathbf{l}_O$  the angular momentum of the mechanism with respect to origin  $O$  of the fixed coordinate frame  $\{x, y, z\}$ . Moreover, if the ground link is denoted by link 1, we can write

$$\mathbf{F}^* = [F_x^* \quad F_y^* \quad F_z^*]^T = -\frac{d}{dt} \sum_{i=2}^N m_i \dot{\mathbf{r}}_i \quad (2)$$

$$\mathbf{M}_O^* = [M_{Ox}^* \quad M_{Oy}^* \quad M_{Oz}^*]^T = -\frac{d}{dt} \sum_{i=2}^N (\tilde{\mathbf{r}}_i m_i \dot{\mathbf{r}}_i + \mathbf{I}^{S_i} \boldsymbol{\omega}_i) \quad (3)$$

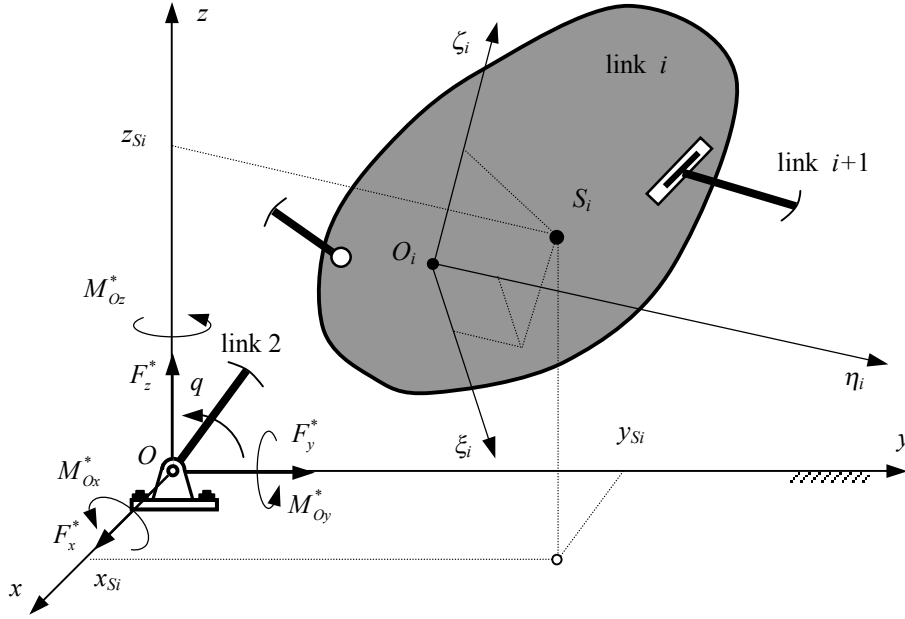


Fig. 1. Coordinate frames and the center of mass of link  $i$

In Eqs. (2) and (3) the following symbols are used:

$m_i$  mass of link  $i$

$N$  number of links

$\mathbf{r}_i$  position vector of the center of mass  $S_i$  in the fixed coordinate frame  $\{x, y, z\}$

$\dot{\mathbf{r}}_i$  velocity vector of  $S_i$  in coordinate frame  $\{x, y, z\}$

$\mathbf{I}^{S_i}$  matrix of the inertia tensor (inertia matrix) of link  $i$  referred to  $S_i$  in coordinate frame  $\{x, y, z\}$

$\boldsymbol{\omega}_i$  angular velocity of link  $i$  with respect to coordinate frame  $\{x, y, z\}$

$\tilde{\mathbf{r}}_i$  3×3 skew-symmetric matrix

$$\mathbf{r}_i = \begin{bmatrix} x_{Si} \\ y_{Si} \\ z_{Si} \end{bmatrix}, \quad \tilde{\mathbf{r}}_i = \begin{bmatrix} 0 & -z_{Si} & y_{Si} \\ z_{Si} & 0 & -x_{Si} \\ -y_{Si} & x_{Si} & 0 \end{bmatrix}. \quad (4)$$

The velocity  $\dot{\mathbf{r}}_i$  and the angular velocity  $\boldsymbol{\omega}_i$  can be expressed in terms of the generalized coordinate  $q$  as

$$\dot{\mathbf{r}}_i = \mathbf{J}_{T_i}(q)\dot{q}, \quad \boldsymbol{\omega}_i = \mathbf{J}_{R_i}(q)\dot{q}, \quad (5)$$

where Jacobian matrices  $\mathbf{J}_{T_i}$  and  $\mathbf{J}_{R_i}$  are

$$\mathbf{J}_{T_i}(q) = \frac{d\mathbf{r}_i}{dq}, \quad \mathbf{J}_{R_i}(q) = \frac{d\boldsymbol{\omega}_i}{d\dot{q}}. \quad (6)$$

Substitution of Eq. (5) into Eqs. (2) and (3) yields the following expressions of the shaking force and the shaking moment

$$\mathbf{F}^* = -\frac{d}{dt} \sum_{i=2}^N m_i \mathbf{J}_{T_i}(q) \dot{q}, \quad (7)$$

$$\mathbf{M}_O^* = -\frac{d}{dt} \left\{ \sum_{i=2}^N \left[ \mathbf{I}^{S_i}(q) \mathbf{J}_{R_i}(q) + m_i \tilde{\mathbf{r}}_i(q) \mathbf{J}_{T_i}(q) \right] \dot{q} \right\}. \quad (8)$$

Eqs. (7) and (8) can be rewritten in the compact form

$$\mathbf{F}^* = -\frac{d}{dt} (\mathbf{m}_T \dot{q}), \quad \mathbf{M}_O^* = -\frac{d}{dt} (\mathbf{m}_R \dot{q}), \quad (9)$$

where the generalized mass matrices  $\mathbf{m}_T$  and  $\mathbf{m}_R$  are column matrices

$$\mathbf{m}_T(q) = \sum_{i=2}^N m_i \mathbf{J}_{T_i}(q), \quad \mathbf{m}_R(q) = \sum_{i=2}^N \left[ \mathbf{I}^{S_i}(q) \mathbf{J}_{R_i}(q) + m_i \tilde{\mathbf{r}}_i(q) \mathbf{J}_{T_i}(q) \right]. \quad (10)$$

Furthermore, the dynamic loads produced by several mechanisms located on the same machine frame and driven by a single drive can be calculated as a summation of the shaking forces and moments of all mechanisms from Eq. (9).

## 2.2 Dynamic load acting on the base from a moving body

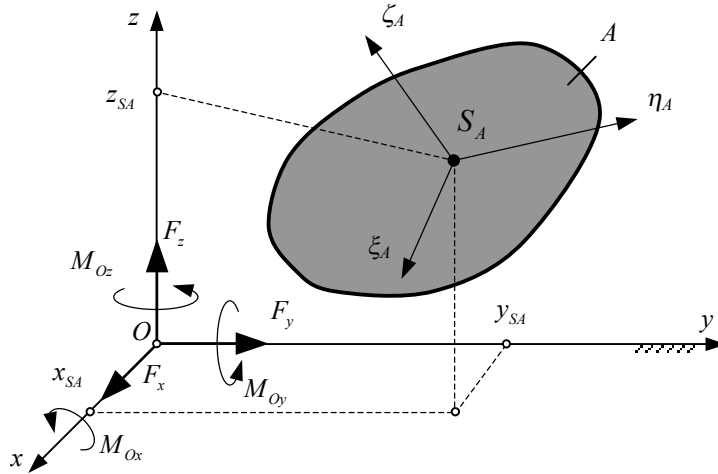


Fig. 2. Coordinate frames and the center of mass of a moving body

The dynamic load acting on the base produced by moving rigid body  $A$  (Fig. 2) can be expressed analogously to Eqs. (2) and (3) in the form

$$\mathbf{F} = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix}^T = -\frac{d}{dt} (m_A \dot{\mathbf{r}}_{SA}) \quad (11)$$

$$\mathbf{M}_O = \begin{bmatrix} M_{Ox} & M_{Oy} & M_{Oz} \end{bmatrix}^T = -\frac{d}{dt} (\tilde{\mathbf{r}}_{SA} m_A \dot{\mathbf{r}}_{SA} + \mathbf{I}_A^S \boldsymbol{\omega}_A), \quad (12)$$

in which

$m_A$  mass of the body

- $\mathbf{r}_{S_A}$  position vector of the center of mass  $S_A$  in the fixed coordinate frame  $\{x, y, z\}$
- $\dot{\mathbf{r}}_{S_A}$  velocity vector of  $S_A$  in coordinate frame  $\{x, y, z\}$
- $\mathbf{I}_{S_A}^S$  matrix of the inertia tensor referred to  $S_A$  in coordinate frame  $\{x, y, z\}$
- $\boldsymbol{\omega}_A$  angular velocity of the body with respect to coordinate frame  $\{x, y, z\}$ .

As can be seen from Eqs. (11) and (12), the dynamic force and moment caused by a spatial rigid body depend on ten inertia parameters of the body and six kinematic parameters (or six independent generalized coordinates) in a coordinate vector  $\mathbf{q}_A$  which describes the position of the body. In our balancing method, the dynamic force  $\mathbf{F}$  and moment  $\mathbf{M}_O$  can be used to fully compensate the shaking force and shaking moment of mechanisms located on the same machine frame. To create a desired time history of dynamic force  $\mathbf{F}$  and moment  $\mathbf{M}_O$  for the purpose of the mass balancing, the motion of the balancing body must be controlled according to a predetermined time history of  $\mathbf{q}_A$ . In practice, this can be realized by means of a drive system with actuators, see also the example in Section 3.

We note that velocity  $\dot{\mathbf{r}}_{S_A}$  and angular velocity  $\boldsymbol{\omega}_A$  can be expressed in terms of velocity vector  $\dot{\mathbf{q}}_A$  as follows

$$\dot{\mathbf{r}}_{S_A} = \mathbf{J}_T(\mathbf{q}_A)\dot{\mathbf{q}}_A, \quad \boldsymbol{\omega}_A = \mathbf{J}_R(\mathbf{q}_A)\dot{\mathbf{q}}_A \quad (13)$$

where Jacobian matrices  $\mathbf{J}_T$  and  $\mathbf{J}_R$  are given by

$$\mathbf{J}_T(\mathbf{q}_A) = \frac{\partial \mathbf{r}_{S_A}}{\partial \mathbf{q}_A}, \quad \mathbf{J}_R(\mathbf{q}_A) = \frac{\partial \boldsymbol{\omega}_A}{\partial \dot{\mathbf{q}}_A}. \quad (14)$$

Eqs. (11) and (12) can then be rewritten in the compact form

$$\mathbf{F} = -\frac{d}{dt}(\mathbf{M}_T\dot{\mathbf{q}}_A), \quad \mathbf{M}_O = -\frac{d}{dt}(\mathbf{M}_R\dot{\mathbf{q}}_A), \quad (15)$$

where the generalized mass matrices  $\mathbf{M}_T$  and  $\mathbf{M}_R$  are

$$\mathbf{M}_T = m_A \mathbf{J}_T(\mathbf{q}_A), \quad \mathbf{M}_R = \mathbf{I}_{S_A}^S \mathbf{J}_R(\mathbf{q}_A) + m_A \tilde{\mathbf{r}}_{S_A} \mathbf{J}_T(\mathbf{q}_A). \quad (16)$$

### 2.3 General balancing conditions

According to on Euler's laws of motion, the dynamic forces acting on the machine frame caused by several mechanisms can be fully balanced through the controlled motion of the balancing body, if the following conditions for the shaking force and the shaking moment are satisfied

$$\mathbf{F}^* + \mathbf{F} = \mathbf{0}, \quad \mathbf{M}_O^* + \mathbf{M}_O = \mathbf{0} \quad (17)$$

Substitution of Eqs. (9) and (14) into Eq. (17) yields the general balancing conditions for the shaking force and shaking moment

$$\mathbf{m}_T(q)\dot{q} + \mathbf{M}_T\dot{\mathbf{q}}_A = \mathbf{0}, \quad \mathbf{m}_R(q)\dot{q} + \mathbf{M}_R\dot{\mathbf{q}}_A = \mathbf{0}. \quad (18)$$

The velocity vector  $\dot{\mathbf{q}}_A$  of the balancing body is then calculated using Eq. (18), which forms a set of six linear algebraic equations in the case of spatial motions or three linear algebraic equations for planar motions. If the inertia parameters of the balancing body in matrices  $\mathbf{M}_T$  and  $\mathbf{M}_R$  are known, then the coordinate vector  $\mathbf{q}_A$  can then be determined by integrating Eq. (18) with respect to time.

### 2.4 Shaking force and shaking moment balancing of planar mechanisms with a balancing body

We consider now the mass balancing problem of planar mechanisms. The velocity relationships for link  $i$  are as follows

$$\dot{\mathbf{r}}_i = [\dot{x}_{Si} \quad \dot{y}_{Si} \quad \dot{z}_{Si}]^T = [x'_{Si} \quad y'_{Si} \quad 0]^T \dot{q}, \quad \boldsymbol{\omega}_i = [0 \quad 0 \quad \dot{\phi}_i]^T = [0 \quad 0 \quad \phi'_i]^T \dot{q}, \quad (19)$$

where the prime represents the derivative with respect to the generalized coordinate  $q$ :  $(\quad)' = \frac{d}{dq}(\quad)$ .

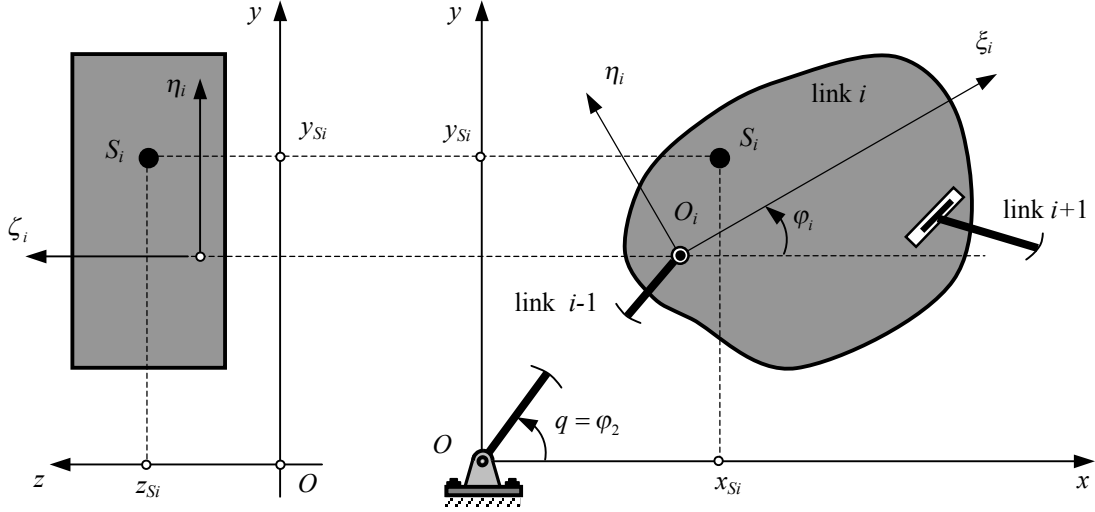


Fig. 3. Coordinate frames of link  $i$  of a planar mechanism (Dresig und Vul'fson, 1989)

Substitution of Eq. (19) into Eq. (6) yields Jacobian matrices

$$\mathbf{J}_{R_i}(q) = [0 \quad 0 \quad \varphi_i']^T, \quad \mathbf{J}_{T_i}(q) = [x'_{Si} \quad y'_{Si} \quad z'_{Si}]^T \quad (20)$$

It is assumed that every link moves in a plane parallel to  $\{x, y\}$ -plane as shown in Fig. 3. This yields  $z_{Si} = \text{const}$  and  $z'_{Si} = 0$ . The inertia matrix  $\mathbf{I}^{S_i}$  of link  $i$  with respect to  $S_i$  in the fixed coordinate frame  $\{x, y, z\}$  can be expressed in the form

$$\mathbf{I}^{S_i} = \mathbf{A}_i \bar{\mathbf{I}}^{S_i} \mathbf{A}_i^T \quad (21)$$

in which  $\bar{\mathbf{I}}^{S_i}$  is the inertia matrix of link  $i$  with respect to the link-fixed coordinate frame  $\{\xi_i, \eta_i, \zeta_i\}$  and  $\mathbf{A}_i$  is the rotation matrix

$$\bar{\mathbf{I}}^{S_i} = \begin{bmatrix} I_{\xi\xi}^S & I_{\xi\eta}^S & I_{\xi\zeta}^S \\ I_{\eta\xi}^S & I_{\eta\eta}^S & I_{\eta\zeta}^S \\ I_{\zeta\xi}^S & I_{\zeta\eta}^S & I_{\zeta\zeta}^S \end{bmatrix}, \quad \mathbf{A}_i = \begin{bmatrix} \cos \varphi_i & -\sin \varphi_i & 0 \\ \sin \varphi_i & \cos \varphi_i & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (22)$$

By substituting Eqs. (19)-(22) into Eq. (10), we obtain the well-known relation for the generalized mass matrices, e.g. (Dresig und Vul'fson, 1989)

$$\mathbf{m}_T = \begin{bmatrix} \sum_{i=2}^N m_i x'_{Si} \\ \sum_{i=2}^N m_i y'_{Si} \\ 0 \end{bmatrix}, \quad \mathbf{m}_R = \begin{bmatrix} \sum_{i=2}^N [-m_i z_{Si} y'_{Si} + (I_{\xi\xi}^S \cos \varphi_i - I_{\eta\xi}^S \sin \varphi_i) \varphi_i'] \\ \sum_{i=2}^N [m_i z_{Si} x'_{Si} + (I_{\xi\xi}^S \sin \varphi_i + I_{\eta\xi}^S \cos \varphi_i) \varphi_i'] \\ \sum_{i=2}^N [m_i (x_{Si} y'_{Si} - y_{Si} x'_{Si}) + I_{\zeta\xi}^S \varphi_i'] \end{bmatrix} \quad (23)$$

When these mass matrices are known, the shaking force and shaking moment of the planar mechanism are then given by Eq. (9). The generalized mass matrices of several parallel mechanisms that are driven by an input shaft can be determined as a summation of the generalized mass matrices of all mechanisms.

In case of planar motion, the balancing body has three degrees of freedom that can be described by two Cartesian coordinates of the center of mass  $S_A$  in  $\{x, y\}$ -plane and rotation angle  $\psi$  about an axis parallel to  $z$ -axis. The coordinate vector  $\mathbf{q}_A$  of the planar balancing body takes the form

$$\mathbf{q}_A = [x_{SA} \quad y_{SA} \quad z_{SA} \quad 0 \quad 0 \quad \psi]^T \quad (24)$$

where  $z_{SA} = \text{const}$ . This yields

$$\dot{\mathbf{q}}_A = [\dot{x}_{SA} \quad \dot{y}_{SA} \quad 0 \quad 0 \quad 0 \quad \dot{\psi}]^T, \quad \boldsymbol{\omega}_A = [0 \quad 0 \quad \dot{\psi}]^T. \quad (25)$$

Using Eq. (14) the Jacobian matrices are given by

$$\mathbf{J}_T(\mathbf{q}_A) = \frac{\partial \mathbf{r}_{SA}}{\partial \mathbf{q}_A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad (26)$$

$$\mathbf{J}_R(\mathbf{q}_A) = \frac{\partial \boldsymbol{\omega}_A}{\partial \mathbf{q}_A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (27)$$

Analogous to Eq. (21), the inertia matrix of the balancing body with respect to the center of mass  $S_A$  in the fixed coordinate frame  $\{x, y, z\}$  has the following form

$$\mathbf{I}_A^S = \mathbf{A} \bar{\mathbf{I}}_A^S \mathbf{A}^T \quad (28)$$

where  $\bar{\mathbf{I}}_A^S$  is the inertia matrix with respect to the center of mass  $S_A$  in the moving coordinate frame  $\{\xi_A, \eta_A, \zeta_A\}$  and  $\mathbf{A}$  is the rotation matrix of the balancing body

$$\bar{\mathbf{I}}_A^S = \begin{bmatrix} I_{\xi\xi}^{SA} & I_{\xi\eta}^{SA} & I_{\xi\zeta}^{SA} \\ I_{\eta\xi}^{SA} & I_{\eta\eta}^{SA} & I_{\eta\zeta}^{SA} \\ I_{\zeta\xi}^{SA} & I_{\zeta\eta}^{SA} & I_{\zeta\zeta}^{SA} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (29)$$

Substitution of Eqs. (26)-(29) into Eq. (16) yields to the generalized mass matrices of the balancing body

$$\mathbf{M}_T = \begin{bmatrix} m_A & 0 & 0 & 0 & 0 & 0 \\ 0 & m_A & 0 & 0 & 0 & 0 \\ 0 & 0 & m_A & 0 & 0 & 0 \end{bmatrix}, \quad (30)$$

$$\mathbf{M}_R = \begin{bmatrix} 0 & -m_A z_{SA} & m_A y_{SA} & 0 & 0 & I_{\xi\zeta}^{SA} \cos\psi - I_{\eta\zeta}^{SA} \sin\psi \\ m_A z_{SA} & 0 & -m_A x_{SA} & 0 & 0 & I_{\xi\zeta}^{SA} \sin\psi + I_{\eta\zeta}^{SA} \cos\psi \\ -m_A y_{SA} & m_A x_{SA} & 0 & 0 & 0 & I_{\xi\zeta}^{SA} \end{bmatrix}. \quad (31)$$

The dynamic force  $\mathbf{F}$  and moment  $\mathbf{M}$  produced by the balancing body moving in a plane parallel with  $\{x, y\}$ -plane are then determined using Eq. (15). Note that moment  $\mathbf{M}$  has not only the component about the  $z$ -axis, but also components about the  $x$ -axis and the  $y$ -axis (VDI-Richtlinie 2149, 2008).

By inserting Eqs. (23), (30) and (31) into Eq. (18), we obtain two balancing conditions for the shaking force

$$\sum_{i=2}^N m_i x'_{Si}(q) \dot{q} + m_A \dot{x}_{SA} = 0, \quad (32)$$

$$\sum_{i=2}^N m_i y'_{Si}(q) \dot{q} + m_A \dot{y}_{SA} = 0, \quad (33)$$

and three balancing conditions for the shaking moment

$$\sum_{i=2}^N \left[ m_i (y_{Si} z'_{Si} - z_{Si} y'_{Si}) + (I_{\xi\xi}^S \cos\varphi_i - I_{\eta\xi}^S \sin\varphi_i) \varphi'_i \right] \dot{q} - m_A z_{SA} \dot{y}_{SA} + (I_{\xi\zeta}^{SA} \cos\psi - I_{\eta\zeta}^{SA} \sin\psi) \dot{\psi} = 0, \quad (34)$$

$$\sum_{i=2}^N \left[ m_i (z_{Si} x'_{Si} - x_{Si} z'_{Si}) + (I_{\xi\xi}^S \sin\varphi_i + I_{\eta\xi}^S \cos\varphi_i) \varphi'_i \right] \dot{q} + m_A z_{SA} \dot{x}_{SA} + (I_{\xi\zeta}^{SA} \sin\psi + I_{\eta\zeta}^{SA} \cos\psi) \dot{\psi} = 0, \quad (35)$$

$$\sum_{i=2}^N \left[ m_i (x_{Si} y'_{Si} - y_{Si} x'_{Si}) + I_{\xi\xi}^S \varphi'_i \right] \dot{q} + m_A x_{SA} \dot{y}_{SA} - m_A y_{SA} \dot{x}_{SA} + I_{\xi\zeta}^{SA} \dot{\psi} = 0. \quad (36)$$

There is no dynamic load transmitted to the base during the motion of the planar mechanism, if the balancing conditions according to Eqs. (32) to (35) are satisfied. In the simple case where the mechanism includes flat links only, i.e.  $z_{Si} = z_{SA} = 0$  and  $I_{\xi\xi i}^S = I_{\eta\eta i}^S = 0$ , these balancing conditions are reduced to

$$\sum_{i=2}^N m_i x'_{Si}(q)\dot{q} + m_A \dot{x}_{SA} = 0, \quad (37)$$

$$\sum_{i=2}^N m_i y'_{Si}(q)\dot{q} + m_A \dot{y}_{SA} = 0, \quad (38)$$

$$\sum_{i=2}^N [m_i (x_{Si} y'_{Si} - y_{Si} x'_{Si}) + I_{\xi\xi i}^S \varphi'_i] \dot{q} + m_A x_{SA} \dot{y}_{SA} - m_A y_{SA} \dot{x}_{SA} + I_{\xi\xi}^S \dot{\psi} = 0. \quad (39)$$

### 3 Numerical Example

In this section, three components of the shaking force and the shaking moment caused by a planar mechanism in the steady-state are given as the time-periodic curves (Fig. 4) to demonstrate the proposed balancing method. These curves can be obtained, for example, from measurements of the position-dependent resultant inertial forces and moments on the machine frame.

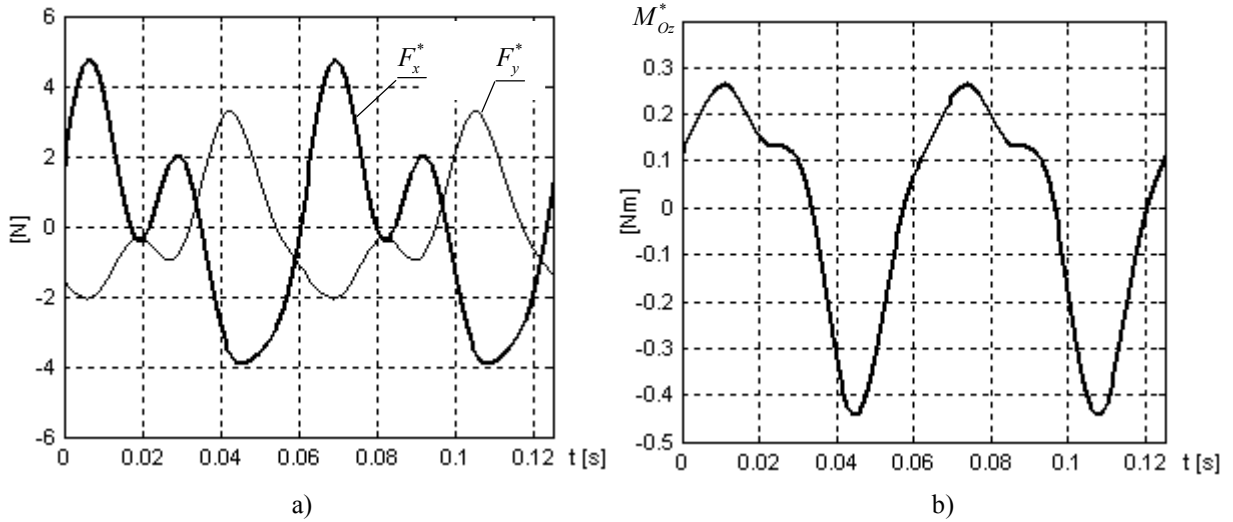


Fig 4. Time curves of a) the shaking force and b) the shaking moment with  $\Omega = 100 \text{ s}^{-1}$

In steady state motion with  $\Omega = \dot{q} = \text{const}$ , the components of the shaking force  $F_x^*$ ,  $F_y^*$  and the shaking moment  $M_{Oz}^*$  according to Eqs. (2) and (3) can be represented by a Fourier series as follows

$$F_x^* = \Omega^2 \sum_k f_{xk} \sin(k\Omega t + \alpha_k), \quad F_y^* = \Omega^2 \sum_k f_{yk} \sin(k\Omega t + \beta_k), \quad M_{Oz}^* = \Omega^2 \sum_k m_{zk} \sin(k\Omega t + \gamma_k), \quad (40)$$

in which Fourier coefficients  $f_{xk}$ ,  $f_{yk}$  and  $m_{zk}$  as well as phase angles  $\alpha_k$ ,  $\beta_k$  and  $\gamma_k$  can be numerically determined by the harmonic analysis of the given time curves (Tab. 1).

$k$	$f_{xk}$ (kgm)	$f_{yk}$ (kgm)	$m_{zk}$ (kgm <sup>2</sup> )	$\alpha_k$ (rad)	$\beta_k$ (rad)	$\gamma_k$ (rad)
1	$3 \times 10^{-4}$	$2 \times 10^{-4}$	$3 \times 10^{-5}$	0,2356	-2,5712	0,2231
2	$2 \times 10^{-4}$	$1,1 \times 10^{-4}$	$1,2 \times 10^{-5}$	1,0016	-1,1209	1,7768
3	$1 \times 10^{-4}$	$5 \times 10^{-5}$	$0,5 \times 10^{-5}$	-0,7398	1,9812	-2,0978

Tab. 1. Fourier coefficients of the shaking force and shaking moment

As mentioned in the last section, the shaking force and moment can be completely compensated through the predetermined motion of a flat balancing body located on the same machine frame. The next step is to calculate the coordinates of the center of mass  $x_{SA}, y_{SA}$  and the rotation angle  $\psi$ .

From the condition for the shaking force balancing

$$m_A \ddot{x}_{SA} = F_x^*, \quad m_A \ddot{y}_{SA} = F_y^*, \quad (41)$$

the coordinates  $x_{SA}, y_{SA}$  are obtained by integrating Eq. (41) twice with respect to time, taking into account the initial conditions

$$x_{SA} = -\frac{1}{m_A} \sum_k \frac{f_{xk}}{k^2} \sin(k\Omega t + \alpha_k) + x_{SA}^0, \quad y_{SA} = -\frac{1}{m_A} \sum_k \frac{f_{yk}}{k^2} \sin(k\Omega t + \beta_k) + y_{SA}^0, \quad (42)$$

in which  $x_{SA}^0$  and  $y_{SA}^0$  represent the position where all velocity components of the balancing body are zero. The use of the balancing condition for the shaking moment yields

$$m_A x_{SA} \ddot{y}_{SA} - m_A y_{SA} \ddot{x}_{SA} + I_A \ddot{\psi} = M_{Oz}^*, \quad (43)$$

where  $I_A = I_{\zeta\zeta}^A$ . By substituting Eq. (42) into Eq. (43), we can represent the obtained result as a Fourier series in the form

$$I_A \ddot{\psi} = \Omega^2 \sum_k m_{\psi k} \sin(k\Omega t + \delta_k), \quad (44)$$

where Fourier coefficients  $m_{\psi k}$  and phase angles  $\delta_k$  depend on coefficients  $f_{xk}, f_{yk}, m_{zk}, \alpha_k, \beta_k, \gamma_k$  as well as  $x_{SA}^0$  and  $y_{SA}^0$ . The rotation angle  $\psi$  are obtained by integrating Eq. (44) twice with respect to time as follows

$$\psi = \frac{1}{I_A} \sum_k \frac{m_{\psi k}}{k^2} \sin(k\Omega t + \delta_k) + \psi_0, \quad (45)$$

where  $\psi_0$  denotes the initial rotation angle of the balancing body.

In addition, a number of parameters such as mass  $m_A$ , inertia moment  $I_A$ , coordinates  $x_{SA}^0, y_{SA}^0$  and angle  $\psi_0$  representing the initial position of the balancing body must be given for the calculation. These parameters are optional. Fig. 5 shows the calculating results for  $x_{SA}, y_{SA}$  and  $\psi$ .

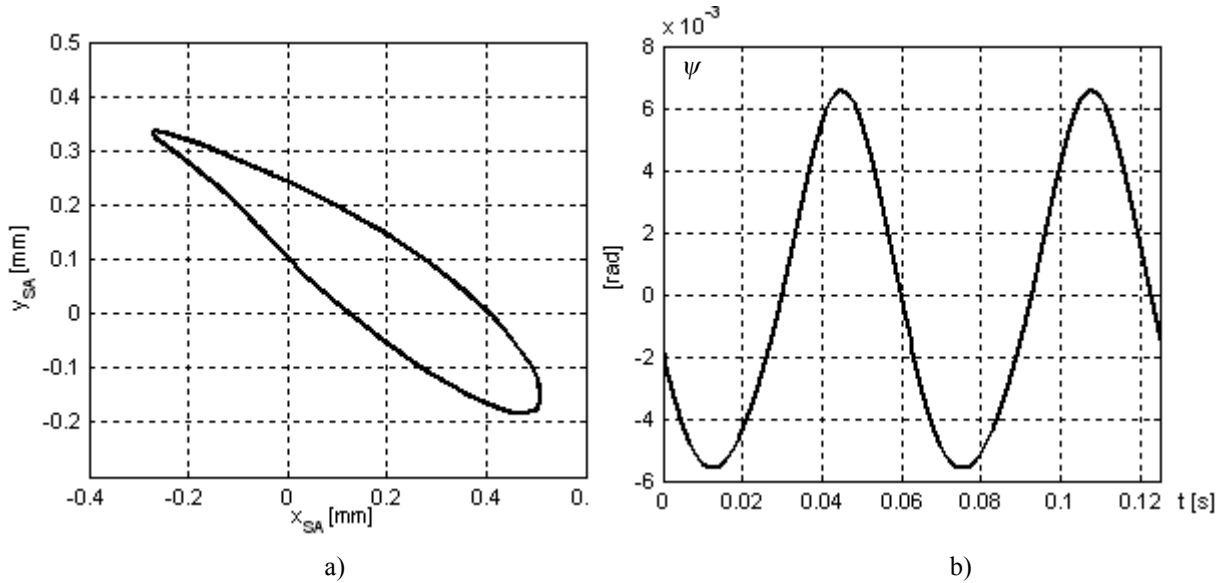


Fig. 5. Calculating results: a) trajectory of the center of mass of the balancing body b) time plot of the rotation angle with  $m_A = 0,8 \text{ kg}$ ,  $I_A = 0,005 \text{ kgm}^2$ ,  $x_A^0 = y_A^0 = 10^{-4} \text{ m}$  and  $\psi_0 = 0$ .



Fig. 6a shows a simple balancing body which are located on the same base with the mechanism. Its geometry is described by length  $a$ , angles  $\alpha$  and  $\beta$ . Assuming that the support reaction forces act on the balancing body at points  $A, B$  and  $C$  as shown in the figure. These forces compose a force vector

$$\mathbf{f}_L = [F_A(t), F_B(t), F_C(t)]^T \quad (46)$$

To control the motion of the balancing body corresponding to the required motion law, we can use the “control coordinates”  $\mathbf{q}_L = [q_A(t), q_B(t), q_C(t)]^T$  which are supposed to have the same direction with the corresponding reaction force (Fig. 6b). The control coordinates have the positive sign when they are directed outwards from the body.

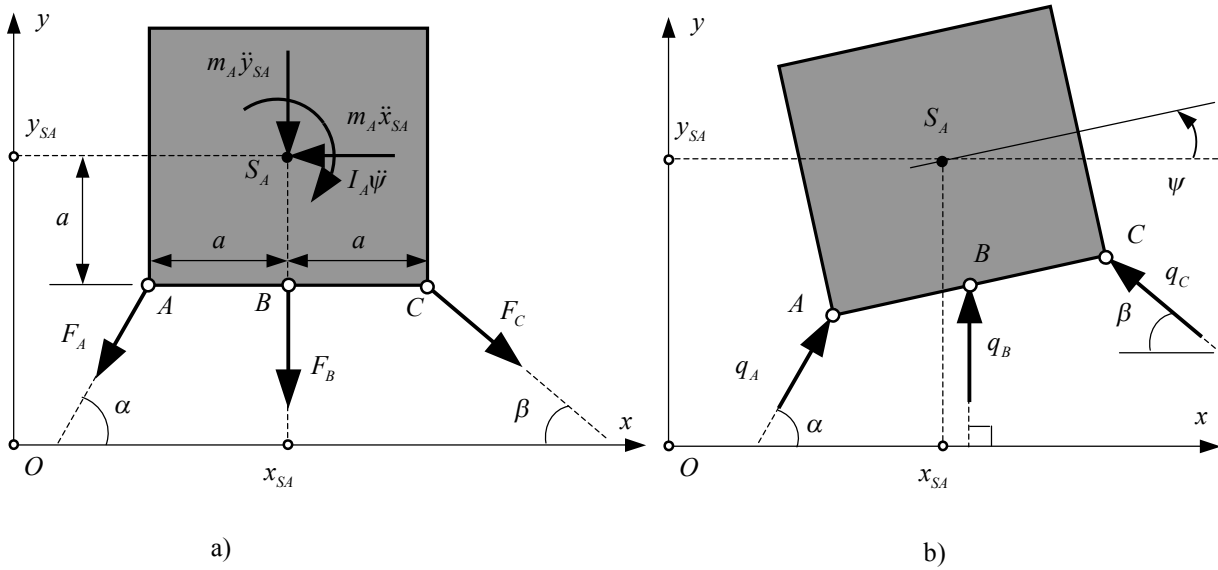


Fig. 6. a) a simple balancing body with reaction forces, b) control coordinates

The support reaction forces  $\mathbf{f}_L$  are linearly dependent on the dynamic loads caused by the balancing body. That leads to the following relationship

$$\mathbf{f}_{S_A} = \mathbf{G}\mathbf{f}_L, \quad (47)$$

where the elements of matrix  $\mathbf{G}$  are geometrical parameters and vector  $\mathbf{f}_{S_A}$  includes three components of the dynamic load with respect to the center of mass  $S_A$

$$\mathbf{f}_{S_A} = [-m_A \ddot{x}_{S_A}, -m_A \ddot{y}_{S_A}, -I_A \ddot{\psi}]^T. \quad (48)$$

It follows

$$\mathbf{f}_L = \mathbf{G}^{-1} \mathbf{f}_{S_A} \quad (49)$$

From Fig. 6a we get

$$\mathbf{G} = \begin{bmatrix} \cos \alpha & 0 & -\cos \beta \\ \sin \alpha & 1 & \sin \beta \\ a(\cos \alpha - \sin \alpha) & 0 & a(\sin \beta - \cos \beta) \end{bmatrix} \quad (50)$$

In case of small motions, the relationship between the control coordinates  $\mathbf{q}_L$  and the coordinate vector  $\mathbf{q}_{S_A} = [x_{S_A}, y_{S_A}, \psi]^T$  is given by

$$\mathbf{q}_L = -\mathbf{G}^T \mathbf{q}_{S_A} \quad (51)$$

Since matrix  $\mathbf{G}$  and coordinate vector  $\mathbf{q}_{Sd}$  are known, the control coordinates can be easily determined using Eq. (51). Fig. 7 shows the calculating results for  $q_A$ ,  $q_B$  and  $q_C$ . If these results are applied to control three linear actuators that drive the balancing body, then inertia effects of the balancing body will compensate the shaking force and shaking moment produced by the mechanism.

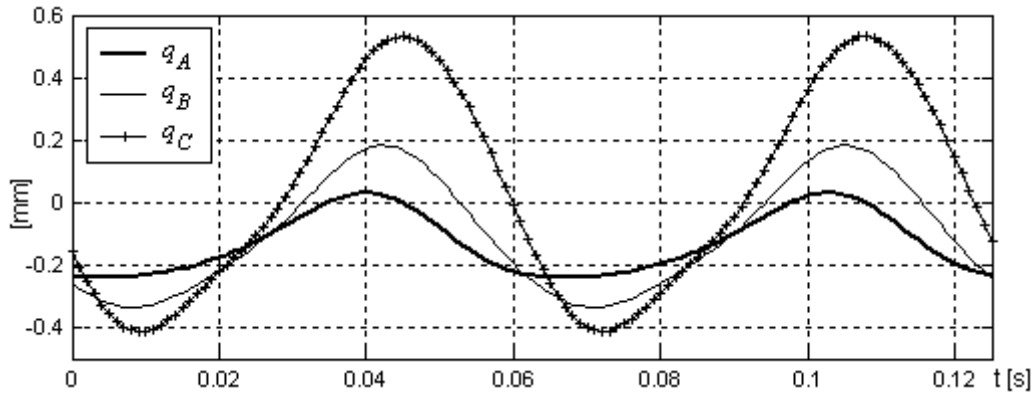


Fig. 7. Control coordinates calculated from Eq. (51) with geometrical parameters:  $\alpha = 70^\circ$ ,  $\beta = 30^\circ$  and  $a = 4 \times 10^{-3}$  m

In performing the foregoing calculations, the drives used to control the motion of the balancing body are assumed to be massless. This assumption can be fulfilled with piezoelectric actuators, so the proposed method can be successfully applied in practice.

#### 4 Conclusions

Dynamic loads in machines and equipments that arise from inertia effects of one or more planar or spatial mechanisms need to be compensated in practice. By using the conventional balancing methods, the cost for the dynamic balancing of multi-link mechanisms is relatively high compared to the initial mechanisms.

By motion control of a single rigid body, any resultant inertia force and moment of several mechanisms can also be fully balanced. This balancing body is installed in the same machine frame with the mechanisms. The required motion of the balancing body is calculated in order that the sum of inertia forces and moments of the mechanisms and the balancing body is zero. As a result, the inertia effects can be completely compensated, so that the machine frame becomes “reactionless”. The required time histories of the three coordinates (plane case: 3 degrees of freedom) or six coordinates (spatial case: 6 degrees of freedom) of the balancing body can be calculated when the kinematic and inertia parameters of the mechanisms or time records of the measured inertia forces and moments are given.

The proposed balancing body is an alternative to the usual balancing mechanisms which require much more design and manufacturing cost. In case of large objects and large motions, the inertia properties of the drives used for the motion control must be taken into account, so that the practical implementation is more difficult in comparison with the case of small motions. It is expected future applications in micro- and nano-technology, where the motion control of the balancing body is possible with piezoelectric actuators to the millimeter range.

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