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On variational updates for non-associative kinematic hardening of Armstrong-Frederick-type

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In this paper, a variational constitutive update for finite strain plasticity theory under consideration of nonlinear kinematic hardening is presented. The proposed scheme permits to compute the unknown state variables such as the plastic part of the deformation gradient, together with the deformation mapping, by means of a fully variational principle. More precisely and in line with the postulate of minimum potential energy, the considered mechanical problem is solely driven by energy minimization. Obviously, the advantages resulting from such a variationally consistent method are manifold. While variational constitutive updates have been successfully derived for relatively simple fully isotropic plasticity models, their extensions to more realistic constitutive laws, particularly to those showing non-associative evolution equations, is highly challenging. In the present paper, a class of finite strain plasticity models characterized by nonlinear kinematic hardening resulting from non-associative evolution equations is recast into an equivalent minimization principle yielding a variational constitutive update. As an important prototype, a variationally consistent Armstrong-Frederick model is discussed.

1 Introduction

One of the most important and very active areas of research in computational mechanics is concerned with efficient implementations of constitutive models. For a comprehensive overview related to computational plasticity, the interested reader is referred to Simo and Hughes (1998); Simo (1998). Considering a certain time interval $[t_n; t_{n+1}]$, the goal of computational plasticity is the determination of all state and history variables $\boldsymbol{X}$ such as the plastic strains at time t_{n+1}, i.e., $\boldsymbol{X}_n \rightarrow \boldsymbol{X}_{n+1}$. The probably best known algorithm for that purpose is the return-mapping scheme, cf. Simo and Hughes (1998); Simo (1998). It consists of two steps. First, the evolution equations and the flow rule characterizing the constitutive model are integrated numerically by employing a backward Euler integration. Subsequently, the resulting set of nonlinear algebraic equations is solved by utilizing Newton's method. Since the backward Euler integration is based on a linear approximation, the final algorithm is first-order accurate as well. Nowadays, the return-mapping scheme is frequently applied and has de facto become a state-of-the-art method, cf. Eterovic and Bathe (1990); Simo (1992); Cuitiño and Ortiz (1992); Dettmer and Reese (2004); Amero (2006); Kuchnicki et al. (2006); Cardoso and Yoon (2008); Hartmann and Bier (2008).

As mentioned before, the return mapping scheme is based on a direct approximation of the differential equations defining the respective constitutive model. Hence, it represents a very general framework and can be applied to the implementation of almost every material model. Though this seems to be an advantage, it is evident that such a numerical formulation does not contain any additional information concerning the underlying physics. Therefore, the method can neither be optimal from a mathematical nor from a physical point of view. For instance, a broad class of constitutive models is characterized by associative evolution equations and flow rules, i.e., they obey the so-called *normality rule*. It is well known that such models can be elegantly derived by employing the postulate of maximum dissipation, cf. Onsager (1931, 1945); Hill (1950); Halphen and Nguyen (1975); Hackl (1997); Yang et al. (2005). They are referred to as *standard dissipative solids*, cf. Halphen and Nguyen (1975). Though the aforementioned underlying variational framework, i.e., maximization of the dissipation, is crucial and implies several consequences, it is not naturally included within the return-mapping scheme.

Numerical implementations directly based on the governing variational structure of constitutive models such as energy minimization or dissipation maximization are referred to as *variational constitutive updates*. This term was originally introduced by Ortiz and co-workers, cf. Ortiz and Stainier (1999); Radovitzky and Ortiz (1999); Ortiz and Repetto (1999). Those authors advocated a method in which every aspect is driven by energy minimization. More precisely and without going too much into details, they proved for a certain class of constitutive models

that minimization of the stress power contains the postulate of maximum dissipation. Furthermore, they proposed a physically and mathematically sound algorithmic formulation simply by discretizing the aforementioned minimization principle. Thus, the resulting implementation inherits the variational structure of the underlying continuous problem, i.e., all state variables, together with the deformation mapping, follow jointly from minimizing a certain energy functional. Clearly, the advantages resulting from this variational scheme such as the existence of a natural metric which can be used for error estimation or the possibility of employing standard optimization algorithms are manifold. It bears emphasis that a similar extremal principle has already been published earlier by Comi and co-workers, cf. Comi et al. (1991); Comi and Perego (1995). However, and in sharp contrast to Ortiz and Stainier (1999); Radovitzky and Ortiz (1999); Ortiz and Repetto (1999), the numerical method proposed by Comi is based on a mixed finite element formulation which does not comply well with the nowadays standard local structure of computational plasticity (the constitutive model is usually enforced only at the integration points). Since the first ideas have been advocated by Ortiz and co-workers, variational constitutive updates have been significantly further elaborated, e.g., in Miehe (2002); Carstensen et al. (2002); Ortiz and Pandolfi (2004); Yang et al. (2005); Fancello et al. (2006); Weinberg et al. (2006); El Sayed et al. (2008); Fancello et al. (2008).

Recently, a general numerical framework for the implementation of standard dissipative solids was given in Mosler and Bruhns (2009a). In principle, it can be applied to any finite strain plasticity model based on normality rules. Unfortunately, it is well known that in some cases, non-associative evolution equations are more realistic. A physically sound framework to deal with such models is given by that of *generalized standard materials* (GSM) Mandel (1972); Lemaitre (1985). In contrast to standard dissipative solids, GSM do not derive from the classical postulate of maximum dissipation. With few exceptions, one presented in Mosler and Bruhns (2009b), it is not clear at all, under which circumstances a modified or extended variational principle exists for GSM. In this paper, a novel extended principle of maximum dissipation is derived and applied to the modeling of nonlinear kinematic hardening of Armstrong-Frederick-type.

The paper is organized as follows: Section 2 and Section 3 are concerned with a concise review of variational constitutive updates for standard dissipative solids. While Section 2 is associated with the fundamentals, an efficient numerical implementation is briefly explained in Section 3. An extended variational principle for nonlinear kinematic hardening of Armstrong-Frederick-type is elaborated in Section 4 and the respective algorithmic formulation is given in Section 5. The symmetry of the resulting tangent operator is discussed in Section 6 and the accuracy of the numerical model is demonstrated in Section 7.

2 Variational constitutive updates for standard dissipative solids - Fundamentals

The variational structure of standard dissipative solids is briefly summarized in this section. Focus is on finite strain plasticity theory. This section follows to a large extend Ortiz and Stainier (1999); Carstensen et al. (2002); Mosler and Bruhns (2009a).

Without going too much into details, it can be shown that the stationarity conditions of the functional

$$\begin{aligned} \tilde{\mathcal{E}}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\Sigma}, \boldsymbol{Q}) &:= \boldsymbol{P} : \dot{\boldsymbol{F}} + J \\ &= \dot{\Psi}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) + \mathcal{D}(\dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}, \boldsymbol{\Sigma}, \boldsymbol{Q}) + J, \end{aligned} \tag{1}$$

define a standard dissipative solid completely. In Eq. (1), $\boldsymbol{\varphi}$ is the deformation mapping, $\boldsymbol{F}^{\mathrm{p}}$ is the plastic part of the deformation gradient, $\boldsymbol{\alpha}$ is a set of strain-like internal variables, $\boldsymbol{\Sigma}$ are the Mandel stress, $\boldsymbol{Q}$ is a set of internal stress-like variables conjugate to $\boldsymbol{\alpha}$, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, Ψ is the Helmholtz energy, $\mathcal{D}$ is the dissipation, the superposed dot denotes the material time derivative and $J = J(\boldsymbol{\Sigma}, \boldsymbol{Q})$ is the characteristic function of the space of admissible stresses. Hence, for admissible stresses $J = 0$ and thus, $\tilde{\mathcal{E}}$ represents the stress power, cf. Ortiz and Stainier (1999); Carstensen et al. (2002). The applicability of the variational principle stat$\tilde{\mathcal{E}}$ is shown here by analyzing exemplarily a variation with respect to the stress-like variables $\boldsymbol{\Sigma}$ and $\boldsymbol{Q}$. More precisely, a maximization of $\tilde{\mathcal{E}}$ with respect to $\boldsymbol{\Sigma}$ and $\boldsymbol{Q}$, i.e., employing the postulate of maximum dissipation, yields the associative flow rule and the respective evolution equation, i.e., $(\dot{\boldsymbol{F}}^{\mathrm{p}} \cdot \boldsymbol{F}^{\mathrm{p}\,-1}, \dot{\boldsymbol{\alpha}}) \in \partial J$. Furthermore, it leads to the reduced functional

$$\mathcal{E}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) := \sup_{\boldsymbol{\Sigma}, \boldsymbol{Q}} \tilde{\mathcal{E}} =: \dot{\Psi}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) + J^*(\dot{\boldsymbol{L}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) \tag{2}$$

with J^* being the Legendre transformation of J. Clearly, physically speaking, $\mathcal{E}$ is still the stress power and J^* represents the dissipation, provided admissible stresses, together with the normality rule, are considered, cf. Ortiz and Stainier (1999); Carstensen et al. (2002); Mosler and Bruhns (2009a).

3 Variational constitutive updates for standard dissipative solids - Numerical implementation

Numerical implementations directly driven by the variational principle

$$\inf \mathcal{E}(\dot{\boldsymbol{\varphi}}, \dot{\boldsymbol{F}}^{\mathrm{p}}, \dot{\boldsymbol{\alpha}}) \tag{3}$$

are referred to as *variational constitutive updates*. They have been advocated by Ortiz and Stainier (1999); Radovitzky and Ortiz (1999); Ortiz and Repetto (1999). Conceptually, they are based on a time discretization of the stress power, i.e.,

$$(\boldsymbol{F}^{\mathrm{p}}_{n+1}, \boldsymbol{\alpha}_{n+1}) = \arg \inf_{\boldsymbol{F}^{\mathrm{p}}_{n+1}, \boldsymbol{\alpha}_{n+1}} \int_{t_n}^{t_{n+1}} \mathcal{E}(\boldsymbol{\varphi}_{n+1}, \boldsymbol{F}^{\mathrm{p}}_{n+1}, \boldsymbol{\alpha}_{n+1}) \, \mathrm{d}t. \tag{4}$$

Although a numerical implementation by means of Eq. (4) seems to be straightforward, it bears emphasis that several nonlinear constraints have to be enforced. Two of those are $\det \boldsymbol{F}^{\mathrm{p}} > 0$ and the plastic constitutive law implied by the flow rule.

An efficient numerical implementation of principle (4) was recently proposed by Mosler and Bruhns (2009a). It is briefly explained in what follows. For the sake of simplicity, a positively homogeneous yield functions ϕ is considered. Combining this with associative evolution equations, it results in the dissipation $\mathcal{D} = \lambda \, Q_0^{\mathrm{eq}}$, with λ being the plastic multiplier and Q_0^{eq} denoting the initial yield stress. For approximating Eq. (4) a first-order time discretization of the type

$$\begin{array}{lcl} \boldsymbol{F}^{\mathrm{p}}_{n+1}(\tilde{\boldsymbol{\Sigma}}, a) & = & \exp\left[a^2 \; \partial_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}}\right] \cdot \boldsymbol{F}^{\mathrm{p}}_n \\ \alpha_{n+1}(a) & = & \alpha_n - a^2 \end{array} \quad , \quad \Delta\lambda = a^2. \tag{5}$$

is adopted. Accordingly, the parameterizations depend on the square root of the plastic multiplier a and so-called *pseudo stresses* $\tilde{\boldsymbol{\Sigma}}$ of Mandel-type. Those stresses are not identical to their physical counterparts, i.e., $\tilde{\boldsymbol{\Sigma}} \neq \boldsymbol{\Sigma}$, but they result by definition in the same flow direction, i.e. $\partial_{\boldsymbol{\Sigma}} \phi|_{\boldsymbol{\Sigma}} = \partial_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}}$. For instance, in case of perfect plasticity of von Mises-type, the set

$$\left\{ \partial_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}} = \frac{\mathrm{Dev}\tilde{\boldsymbol{\Sigma}}}{||\mathrm{Dev}\tilde{\boldsymbol{\Sigma}}||} \;\middle|\; \tilde{\boldsymbol{\Sigma}} \in \mathbb{R}^{3\times 3} \right\} \tag{6}$$

spans the space of all deviatoric unit tensors and hence, the physical constraints associated with von Mises plasticity theory are naturally included. Inserting time discretizations (5) into Eq. (4), yields finally the variational principle

$$\boldsymbol{X} = \arg \inf_{\boldsymbol{X}} I_{\mathrm{inc}}(\boldsymbol{X}), \quad \text{with} \quad I_{\mathrm{inc}} = \int_{t_n}^{t_{n+1}} \mathcal{E} \; \mathrm{d}t = \Psi_{n+1}(\boldsymbol{X}) - \Psi_n + Q_0^{\mathrm{eq}} \, \Delta\lambda, \quad \text{and} \quad \boldsymbol{X} = [\tilde{\boldsymbol{\Sigma}}, a]. \tag{7}$$

As shown in Mosler and Bruhns (2009b), minimization principle (7) is a first-order approximation of the underlying standard dissipative solid.

4 Variational constitutive updates for non-linear kinematic hardening of Armstrong-Frederick-type - Fundamentals

In this section, a variational framework for nonlinear kinematic hardening of Armstrong-Frederick-type is elaborated. In contrast to standard dissipative solids, nonlinear kinematic hardening does no derive from the classical postulate of maximum dissipation, i.e., the evolution equations are non-associative. Hence, an extended variational principle has to be considered. The method advocated here is similar to that recently given in Mosler and Bruhns (2009b). The underlying idea is directly related to the novel parameterization of the flow rule, as utilized in the previous section. More precisely, re-writing Eq. (7) into the slightly more general form

$$\boldsymbol{X} = \arg \inf_{\boldsymbol{X}} I_{\mathrm{inc}}(\boldsymbol{X}), \quad \text{with} \quad I_{\mathrm{inc}} = \int_{t_n}^{t_{n+1}} \mathcal{E} \; \mathrm{d}t = \Psi_{n+1}(\boldsymbol{X}) - \Psi_n + \int_{t_n}^{t_{n+1}} \mathcal{D}(\tilde{\boldsymbol{\Sigma}}, a) \; \mathrm{d}t, \quad \text{with} \quad \boldsymbol{X} = [\tilde{\boldsymbol{\Sigma}}, a], \tag{8}$$

it is evident that the stress power depends on the evolution equations and the flow rule as well as on the dissipation. If the flow direction is parameterized by using the pseudo-stresses $\tilde{\boldsymbol{\Sigma}}$, it is a priori enforced, independently of the

considered dissipation. As a result, $\mathcal{D}$ can be chosen differently, i.e., in a non-associative manner. Consequently, minimization principle (8) gives enough flexibility also for generalized standard materials. It bears emphasis that in this case, consistency of the variational method has to be proved explicitly.

In what follows, a finite strain plasticity model for non-linear kinematic hardening of the type

$$\Psi(\boldsymbol{F}, \boldsymbol{F}^{\mathrm{p}}, \boldsymbol{\alpha}) = \Psi^{\mathrm{e}}(\boldsymbol{C}^{\mathrm{e}}) + \Psi^{\mathrm{p}}(\boldsymbol{\alpha}), \quad \Psi^{\mathrm{p}}(\boldsymbol{\alpha}) = \frac{1}{2}\, c\, \boldsymbol{\alpha} : \boldsymbol{\alpha} \tag{9}$$

$$\phi = ||\mathrm{dev}(\boldsymbol{\Sigma}) - \boldsymbol{Q}|| - Q_0^{\mathrm{eq}}, \quad \boldsymbol{Q} := -\partial_{\boldsymbol{\alpha}} \Psi = -c\, \boldsymbol{\alpha} \tag{10}$$

$$\boldsymbol{L}^{\mathrm{p}} := \dot{\boldsymbol{F}}^{\mathrm{p}} \cdot (\boldsymbol{F}^{\mathrm{p}})^{-1} = \lambda\, \partial_{\boldsymbol{\Sigma}} \phi, \quad \dot{\boldsymbol{\alpha}} = -\boldsymbol{L}^{\mathrm{p}} - \lambda\, b\, \boldsymbol{\alpha} \neq \lambda\, \partial_{\boldsymbol{Q}} \phi = -\boldsymbol{L}^{\mathrm{p}} \tag{11}$$

is considered. Here, Ψ^{e}, Ψ^{p}, ϕ, $\boldsymbol{C}^{\mathrm{e}} := (\boldsymbol{F}^{\mathrm{e}})^T \cdot \boldsymbol{F}^{\mathrm{e}}$, $\boldsymbol{\alpha}$, $\boldsymbol{Q}$ are the elastic part of the Helmholtz energy, an additional part corresponding to plastic work, the yield function, the elastic right Cauchy-Green tensor, a tensor-valued strain-like internal variable and its stress-like conjugate variable being the backstress, respectively. The material parameters c and b denote a hardening coefficient and a variable defining the saturation of $\boldsymbol{\alpha}$, i.e., $||\boldsymbol{\alpha}|| \to 1/b$. Accordingly, the backstress tensor converges eventually to $||\boldsymbol{Q}|| \to c/b$. As evident from Eq. $(11)_2$, the evolution equation governing $\boldsymbol{\alpha}$ is non-associative. Frequently, by using the covariance of Ψ^{p}, this equation is recast by considering a certain objective time derivative of $\boldsymbol{\alpha}$.

According to Eq. (7), the extended variational principle depends on the dissipation. A straightforward transformation yields

$$\mathcal{D} = \boldsymbol{\Sigma} : \boldsymbol{L}^{\mathrm{p}} + \boldsymbol{Q} : \dot{\boldsymbol{\alpha}} = \lambda\, Q_0^{\mathrm{eq}} + \lambda\, c\, b\, \boldsymbol{\alpha} : \boldsymbol{\alpha} \geq 0. \tag{12}$$

In line with Eq. (5) the evolution equations (11) are enforced by using a parameterization in terms of the pseudo stresses $\tilde{\boldsymbol{\Sigma}}$, cf. Mosler and Bruhns (2009b); Mosler and Cirak (2009), i.e.,

$$\boldsymbol{L}^{\mathrm{p}} := \dot{\boldsymbol{F}}^{\mathrm{p}} \cdot \boldsymbol{F}^{\mathrm{p}\,-1} = \lambda\, \partial_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}}, \quad \dot{\boldsymbol{\alpha}} = -\lambda\, \partial_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}} - \lambda\, b\, \boldsymbol{\alpha}. \tag{13}$$

Inserting these equations into the stress power, together with the dissipation (12), leads to

$$\begin{aligned} \mathcal{E}(\dot{\boldsymbol{F}}, \tilde{\boldsymbol{\Sigma}}, \lambda) = \dot{\Psi} + \mathcal{D} &= \partial_{\boldsymbol{F}} \Psi : \dot{\boldsymbol{F}} - \left(\boldsymbol{F}^{\mathrm{e}T} \cdot \frac{\partial \Psi^{\mathrm{e}}}{\partial \boldsymbol{F}^{\mathrm{e}}} \right) : \boldsymbol{L}^{\mathrm{p}} + \lambda\, \boldsymbol{Q} : \boldsymbol{L}^{\mathrm{p}} + \lambda\, Q_0^{\mathrm{eq}} \\ &= \partial_{\boldsymbol{F}} \Psi : \dot{\boldsymbol{F}} - \lambda\, \phi. \end{aligned} \tag{14}$$

Consequently, the stationarity condition with respect to the internal variables reads

$$\begin{aligned} \partial_{\lambda} \mathcal{E} &= -\phi = 0 \\ \partial_{\tilde{\boldsymbol{\Sigma}}} \mathcal{E} &= (\boldsymbol{\Sigma} - \boldsymbol{Q}) : \partial^2_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}} = \boldsymbol{0}. \end{aligned} \tag{15}$$

Thus, the minimization principle $\inf \mathcal{E}$ enforces the yield function ($\phi \leq 0$) and for plastic deformations, the flow direction has to fulfill the constraint $(15)_2$. As already shown in Mosler and Bruhns (2009b), constraint $(15)_2$ represents a compatibility condition between the pseudo stresses an their physical counterparts and is consistent with the underlying flow rule. As a result, the proposed variational principle is indeed equivalent to the constitutive model summarized in Eqs. (9)–(11). Furthermore, the stress response is naturally included as well. More precisely, the first Piola-Kirchhoff stress tensor is given by

$$\boldsymbol{P} = \partial_{\dot{\boldsymbol{F}}} \mathcal{E}. \tag{16}$$

5 Variational constitutive updates for non-linear kinematic hardening of Armstrong-Frederick-type - Numerical implementation

In Section 3, a variational constitutive update for standard dissipative solids (sds) was discussed. It is based on minimizing the integrated stress power. For a priori enforcing the constraint corresponding to the flow direction, a novel parameterization in terms of pseudo stresses was utilized. Though the variational re-formulation of the Armstrong-Frederick model as elaborated in the previous section does not fall into the range of sds, it relies on the same concepts. Hence, a numerical implementation completely in line with that presented in Section 3 can be adopted. Consequently, a backward Euler approximation is applied first, i.e.,

$$\begin{aligned} \boldsymbol{F}^{\mathrm{p}}_{n+1}(\tilde{\boldsymbol{\Sigma}}, a) &= \exp\left[a^2\ \partial_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}}\right] \cdot \boldsymbol{F}^{\mathrm{p}}_n \\ \boldsymbol{\alpha}_{n+1}(\tilde{\boldsymbol{\Sigma}}, a) &= \frac{\boldsymbol{\alpha}_n - a^2\, \partial_{\boldsymbol{\Sigma}} \phi|_{\tilde{\boldsymbol{\Sigma}}}}{1 + a^2\, b} \\ \int_{t_n}^{t_{n+1}} \mathcal{D}\ \mathrm{d}t(\tilde{\boldsymbol{\Sigma}}, a) &= a^2\, Q_0^{\mathrm{eq}} + a^2\, c\, b\, ||\boldsymbol{\alpha}_{n+1}||^2. \end{aligned} \tag{17}$$

Substituting Eqs. (17) into the stress power yields finally

$$I_{\mathrm{inc}}(\tilde{\boldsymbol{\Sigma}}, a) = \Psi_{n+1}(\tilde{\boldsymbol{\Sigma}}, a) - \Psi_n + a^2\, Q_0^{\mathrm{eq}} + a^2\, c\, b\, ||\boldsymbol{\alpha}_{n+1}||^2 \tag{18}$$

and the update of the internal variables follows from the variational principle

$$(\tilde{\boldsymbol{\Sigma}}, a) = \arg\inf I_{\mathrm{inc}}(\tilde{\boldsymbol{\Sigma}}, a), \tag{19}$$

while the stress response is given by

$$\boldsymbol{P} = \partial_{\boldsymbol{F}} \inf I_{\mathrm{inc}}(\tilde{\boldsymbol{\Sigma}}, a). \tag{20}$$

Since the applied backward Euler integrations (and the approximation by using the exponential map) are consistent first-order schemes, the resulting approximation of the stress power is consistent as well. Hence, in the limiting case $\Delta t \to 0$ the numerical scheme converges to the continuous problem (12). Further details are omitted. They can be found in Mosler and Bruhns (2009a,b).

6 A comment on the symmetry of the algorithmic tangent operator

The proposed variational constitutive update is defined by means of the principle

$$\inf_{\boldsymbol{F},\boldsymbol{X}} I_{\mathrm{inc}}, \quad \text{with} \quad I_{\mathrm{inc}} = I_{\mathrm{inc}}(\boldsymbol{F}, \boldsymbol{X}, \boldsymbol{B}). \tag{21}$$

Here, $\boldsymbol{F}$ is the deformation gradient at time t_{n+1}, $\boldsymbol{X}$ is a vector containing all unknowns and $\boldsymbol{B}$ is a vector consisting of all internal variables at time t_n (including the plastic part of the deformation gradient), i.e.,

$$\boldsymbol{X} := \{\tilde{\boldsymbol{\Sigma}}, \Delta\lambda\}, \quad \boldsymbol{B} := \{\boldsymbol{F}_n^{\mathrm{p}}, \boldsymbol{\alpha}_n\}. \tag{22}$$

In contrast to the constitutive updates presented before, I_{inc} is now the global stress power of the structure including additional work parts due to external forces. In line with classical computational plasticity theory (return-mapping scheme, cf. Simo and Hughes (1998)), Problem (21) is usually solved in a hierarchical manner. More precisely, at the integration point level, the local problem

$$\boldsymbol{X} = \arg\inf_{\boldsymbol{X}} I_{\mathrm{inc}}\Big|_{\boldsymbol{F}=\mathrm{const}}, \quad \boldsymbol{P} = \partial_{\boldsymbol{F}} I_{\mathrm{inc}} \tag{23}$$

is computed first and subsequently, the deformation mapping follows from

$$\inf_{\boldsymbol{F}} I_{\mathrm{inc}}\Big|_{\boldsymbol{X}=\mathrm{const}}. \tag{24}$$

Analogously to the return-mapping scheme, Eq. (23) represents the update algorithm of the stresses and that of the internal variables, while Eq. (24) is equivalent to the global equilibrium condition. Hence, if Problem (24) is to be solved by employing Newton's method showing an asymptotic quadratic convergence, the linearization of the stress update algorithm (23) is required. For that purpose, minimization Problem (23) is linearized at a converged solution yielding

$$\mathrm{d}\left(\partial_{\boldsymbol{X}} I_{\mathrm{inc}}\right) = \boldsymbol{0} \quad \Rightarrow \quad \mathrm{d}\boldsymbol{X} = -\left[\partial^2_{\boldsymbol{X}\boldsymbol{X}} I_{\mathrm{inc}}\right]^{-1} \cdot \partial^2_{\boldsymbol{X}\boldsymbol{F}} I_{\mathrm{inc}} : \mathrm{d}\boldsymbol{F}. \tag{25}$$

Inserting this equation into the linearization of the stress tensor $\boldsymbol{P} = \partial_{\boldsymbol{F}} I_{\mathrm{inc}}$, leads finally to

$$\mathrm{d}\boldsymbol{P} = \underbrace{\left\{\partial^2_{\boldsymbol{F}\boldsymbol{F}} I_{\mathrm{inc}} - \partial^2_{\boldsymbol{F}\boldsymbol{X}} I_{\mathrm{inc}} \cdot \left[\partial^2_{\boldsymbol{X}\boldsymbol{X}} I_{\mathrm{inc}}\right]^{-1} \cdot \partial^2_{\boldsymbol{X}\boldsymbol{F}} I_{\mathrm{inc}}\right\}}_{:=\,\mathbb{A}} : \mathrm{d}\boldsymbol{F}. \tag{26}$$

Accordingly and as expected for a variational method, the tangent operator $\mathbb{A}$ shows the major symmetry, i.e.,

$$\mathbb{A}_{ijkl} = \mathbb{A}_{klij}. \tag{27}$$

Clearly, $\mathbb{A}$ represents the so-called algorithmic tangent. It is noteworthy that this symmetry condition is fulfilled independently of the employed consistent time derivative. This is in sharp contrast to the classical return-mapping scheme characterized by a non-symmetric algorithmic tangent operator (for Armstrong-Frederick-type nonlinear kinematic hardening). It bears emphasis that the symmetry of the algorithmic tangent corresponding to the advocated variational constitutive update is completely in line with the symmetry of the underlying continuum tangent. A positive side effect of this symmetry is that the novel update is more efficient from a computational point of view. Further details about the symmetry of the tangent operator are beyond the scope of the present paper. They will be analyzed in a forthcoming work.

shear modulus:	$\mu = 80000$ Mpa
bulk modulus:	$K = 173333$ Mpa
yield stress:	$Q_0^{\text{eq}} = \sqrt{2/3}\ 300$ Mpa
hardening modulus:	$c = 1900$ Mpa
saturation parameter:	$b = 8.5$

Table 1: Material parameters employed for the simple tension and the simple shear test, cf. Dettmer and Reese (2004).

7 Numerical examples

In the present section, the accuracy of the proposed implementation is compared to that of other algorithmic formulations based on the return-mapping scheme. For that purpose, two simple benchmarks are considered: the simple tension and the simple shear test. Those problems have already been previously numerically analyzed for nonlinear kinematic hardening in Dettmer and Reese (2004). In the cited paper, a classical return-mapping algorithm has been applied to different finite strain models of Armstrong-Frederick-type. Following Dettmer and Reese (2004), the material parameters characterizing a mild steel CK15 are chosen. They are summarized in Tab. 1.

According to Fig. 1, the variationally consistent update predicts almost the same mechanical response as the models analyzed in Dettmer and Reese (2004).

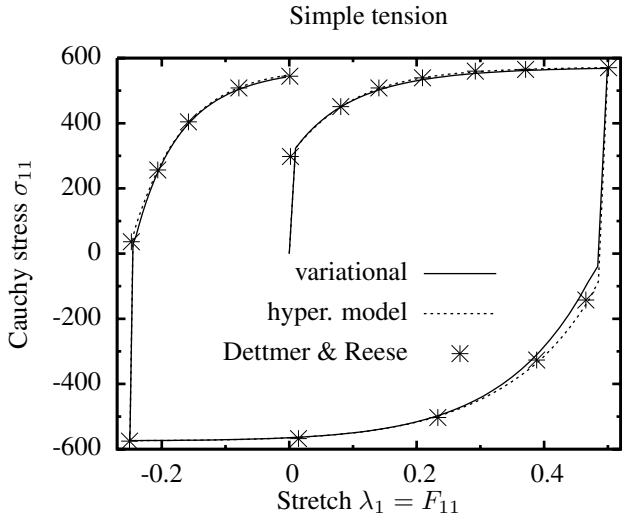

Figure 1: Uniaxial tension test: Material parameters have been chosen according to Dettmer and Reese (2004).

The results obtained from the numerical analyses of the simple shear test are summarized in Fig. 2. Again an excellent agreement between the different models is evident. It is even more pronounced than for the simple tension test. Such an observation has already been made previously in Dettmer and Reese (2004). Clearly, although all models led to almost identical results for the simple tension and the simple shear test, this cannot be guaranteed and is not expected for more general loading cases. Only for infinitesimal deformations, all finite strain plasticity models are equivalent.

8 Conclusions

In this paper, a novel numerical implementation of non-linear kinematic hardening of Armstrong-Frederick-type at finite strains was presented. In contrast to previously published models, the advocated algorithmic framework is fully variational. More precisely, all unknown state variables and the deformation mapping follow jointly from minimizing the stress power. The cornerstone and the starting point of the novel variational constitutive update is an extended principle of maximum dissipation. Roughly speaking and in line with standard dissipative solids which are characterized by associative evolution equations, the aforementioned extended principle is still related

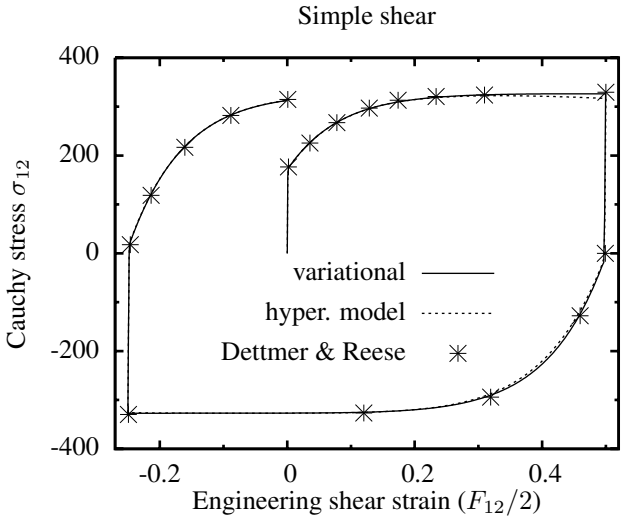

Figure 2: Simple shear test: Material parameters have been chosen according to Dettmer and Reese (2004).

to the postulate of maximum dissipation. However and in sharp contrast to the classical approach, non-associative evolution equations are enforced a priori. By employing a suitable parameterization of the flow rule and the evolution equations, the respective constrained optimization problem can be recast into an equivalent unconstrained minimization problem. This concept was applied to a finite strain plasticity model of Armstrong-Frederick-type. Benchmark problems demonstrated the applicability as well as the accuracy of the algorithmic formulations. It is noteworthy that due to the underlying variational structure of the algorithm, the resulting algorithmic tangent operator is indeed symmetric. This is not the case for standard numerical formulations such as for the return-mapping scheme. Clearly, this symmetry property leads to a more efficient numerical implementation.

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