An Introduction of Internal Constraints in a Natural Way

Notations: let $Sym$ be the set of all symmetric tensors of second order, $Sym^+$ the positive-definite ones in $Sym$, $F$ the deformation gradient, $C = F^T \cdot F \in Sym^+$ the right Cauchy-Green tensor, $T \in Sym$ the stress-tensor, and $S = JF^{-1} \cdot T \cdot F^{-1} \in Sym$ the second Piola-Kirchhoff stress-tensor, with $J = \text{det}(F)$.

According to the classical theory of internal constraints [Truesdell/Noll p. 70] two assumptions are made:
1. There are restrictions upon the possible deformations $C$ of the material element, such that a scalar valued function equals zero for all possible deformations:
   \[ \lambda(C) = 0. \]
2. Principle of determinism for simple materials subject to internal constraints:
   The stress is determined by the history of the deformation only to within an additive part that does no work in any possible motion satisfying the constraints.

As a consequence, this part, the reaction stress, is given by
   \[ S_R = \frac{d\lambda(C)}{dC}. \]

This theory has two shortcomings:
- the form of the constraint is not general enough;
- the principle of determinism has no strict physical substantiation.

Therefore the aim of this paper is to introduce constraints in a natural way. At first I will demonstrate this by a simple example: a fibre-reinforced material may have the property, that the strain in the fibre-direction is caused by directly larger stresses than in any other direction. If the applied stresses are not of this magnitude, it seems to be reasonable to make the idealization of unextensibility in the fibre-direction. We set the stiffness infinitely large, so that finite stresses do not cause any strain in fibre-direction. We interpret those stresses that do not cause any deformations whatever their magnitude is, as reaction stresses. That is, we conceive a material with internal constraints as being the limit of material elements which become infinitely stiff in certain directions. Of course, the properties of these limit-elements depend on the class of materials under consideration. For simplicity I demonstrate this procedure within some subclasses of elastic materials. Let
   \[ S = \delta(C) \]
be the response function of a hyper-elastic material, so that
   \[ dS = \frac{d\delta(C)}{dC} \cdot dC = C \cdot dC, \]
where the 4th-ordered elasticity-tensor $C$ is symmetric. At any fixed deformation $C$ we have the spectral representation
   \[ C = c^i E_i \otimes E_i \]
with six propernumbers $c^i$ and six propertensors $E_i$ which form an orthogonal basis in $Sym$. "Infinite stiffness" means that at least one propernumber, say $c^a$, becomes infinitely large. As a result of the assumption of finite stress increments we obtain the constraint
   \[ E_a \cdot dC = 0, \]
and, if integrable, the classical form
   \[ \lambda(C) = 0. \]
If we write $a^\alpha$ for identical direction in the six-dimensional space $Sym$, we obtain for the reaction-stresses
   \[ S_R \propto C \cdot E_a \propto E_a \]
i.e. those stresses, that do not cause any deformation within the limit as $c^a$ becomes infinite. For the power of the reaction stresses we get
   \[ S_R \cdot dC = 0 \]
for all possible deformation increments $dC \in Sym$ obeying the constraint equation; i.e. the reaction-stresses of hyper-elastic materials with internal constraints do not work in any possible motion. We can construct six independent constraints due to the six propertodirections of the elasticity tensor. If applied simultaneously, these result in the rigid body as being the limit of a hyper-elastic body.

Let us next investigate a class of elastic materials that is not hyper-elastic, but allows the following representation of the elasticity tensor at any $C$:
   \[ C = c^i E_i \otimes E^i \]
where $c^i$ again are the six propernumbers, $E_i$ six right propertensors and $E^i$ six left propertensors, so that
   \[ E_i \cdot E^j = \delta^j_i \]
Using the analogous procedure as above we obtain the constraint equation
\[ dC \cdot E^u = 0 \]
and for the reaction-stresses
\[ S^u \propto E^u. \]
Now it is possible for the work of the reaction-stresses
\[ S^u \cdot dC \]
to not vanish; i.e. the reactions-stresses may work during possible motions within this class of materials.

If we impose more than one constraint the space of the reaction-stresses is simply the one which is spanned by the respective propertensors \( E_w \).

By slightly generalizing this procedure we can also obtain unilateral constraints and those that are deformation dependent (non-steady).

If we consider anelastic materials, we may introduce very unusual non-classical constraints imposed on the deformation processes of the material. For example: the viscosity of a fluid of differential type may become infinitely large at a certain limit of shear velocity. We then obtain shear stresses for the reactions, that — of course — do work.

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Reference
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